Lecture 2: What is Proof?

Math 295

08/26/16
Evolution of Proof

Proof, a relatively “new” idea

Modern mathematics could not be supported at its foundation, nor construct the “top floor” without precise statements

Intuition works up to a point
Evolution of Proof

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Four “Stages”:
Evolution of Proof

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Modern mathematics could not be supported at its foundation, nor construct the “top floor” without *precise statements*

Intuition works up to a point

Four “Stages”:

- just answers—volumes, areas, lengths, angles, relationships; practical
- demonstration (intuition)—that it works, calculation, formulae
- verification/justification—plausible picture, description, analogy, invoke the gods
- modern proof
Right pyramidal frustum
Egyptians: Practical Questions

Right pyramidal frustum

\[ V = \frac{h}{3} (a^2 + ab + b^2) \]

Known in 1850 BCE.

Requires calculus for proof of derivation
Right pyramidal frustum

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“Answer”: No record of derivation or proof.
Euclid: The Outlier

Greeks: Geometry, trigonometry, “forms”, idealizations
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- Triangles, rectangles, circles (building/construction)
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- Precursors: Thales (600 BCE), Pythagoras and Pythagoreans (550 BCE), Eudoxus (400 BCE)—calculations and declarations, “theorems”
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- Axioms, definitions, theorem, proof
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- Axioms, definitions, theorem, proof
- “Demonstration” and “Justification”
- But where did the proofs come from?
Dark Ages
Renaissance Number Theorists

- Gerolamo Cardano (1501–1557) and Niccolo Tartaglia (1500–1576)
- Era of “the master”
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- “Demonstration” and “Verification”: I have the answer, I can check these cases.
Calculus Era I

• Johannes Kepler (1571–1630): Proof, statistical methods

• Isaac Barrow (1630–1667): Tangent problem, discovered the fundamental theorem of calculus, derive solutions to physical problems

• Isaac Newton (1642–1727): Student of Barrow’s, sloppily axiomatize some ideas in calculus to derive equations and solutions to difficult problems; motivated by “infinitesimal change” in physical problems

• Gottfried Leibniz (1646–1716): Philosopher and polymath, looser notation and better intuition provided the backbone of modern analysis
Calculus Era II

Ideas
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- infinitely small
- infinitely big
- “infinitesimals”, “limits”
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Solutions to big problems (e.g., the area problem, the tangent problem, calculus of variations)

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Example: The Basel problem
Golden Era: 18th and 19th Century

Moving towards modern proof
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Moving towards modern proof

- **Joseph Louis Lagrange (1736–1813):** Move from blind formal manipulation and intuition, Taylor polynomials

- **Karl Friedrich Gauss (1777–1855):** Adequate (modern) analysis of series

- **August Louis Cauchy (1789–1857):** French counterpart to Gauss; limits, continuity, differentiation, and definite integral (modern), complex calculus

- **Karl Weierstrass (1815–1897):** Topology, foundations, real numbers, "real analysis" including "$\varepsilon$–$\delta$"
David Hilbert (1862–1943):

- Last conversant mathematician of entire subject
- Wanted *axiomatic* and *consistent* formulation of set theory
- Formalism: mathematics is manipulation of symbols according to agreed upon formal rules
- Hilbert’s Program: Formal system which is complete, consistent, decidable
- Hilbert’s 23 Problems at the International Congress in 1900
- Axiomatic and consistent construction of the integers
- “What are the rules?” and ”What is the context?”
Henri Poincare (1854–1912): Great mathematician; intuition—opposite Hilbert and Russell; mathematics fundamentally distinct from logic; proof has intuitive steps, not purely mechanical

Nicolas Bourbaki (1930’s): pseudonym for French mathematicians; encyclopedists: consistent, rigorous treatment of all common mathematics; no pictures, no intuition

Bertrand Russell (1872–1970): foundations; reducing mathematics to its most elementary “grains”; logicism; constructing the integers

Kurt Godel (1906–1978): Incompleteness; destruction of Hilbert’s Program and Russell/Whitehead Foundationalism/logicism
In 1931 he proved:

For any computable axiomatic system that is powerful enough to describe the natural numbers/arithmetic (e.g. the Peano axioms or ZFC set theory), that:

- If the formal system is consistent, it cannot be complete.
- The consistency of the axioms cannot be proven within the system.

These theorems ended attempts by many of those above to find a "small" set of consistent axioms sufficient for all of modern mathematics.
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Russell’s Two Directions and Weirdness

WEIRDNESS

← Foundations ← Seemingly Self-Evident Mathematical Facts → The Infinite →

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Modern Mathematics: walking on egg shells
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Modern Mathematics: walking on egg shells

Proof in this class:

- Pretty axiomatic, basic
- We will not prove everything, and we will define many things intuitively
- Take many basic mathematical facts as given
- May take somethings for granted and come back and prove them later
Axiom

1 Definition versus Theorem
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- **Hypothesis(es)/ Conclusion**
Proof

A sequence of logically permissible steps connecting the hypothesis to the conclusion—uses rules of inference and logic

Many types of proof, many proof techniques

Lemma, Corollary, Proposition

A Lemma is a "small" statement which only requires a "small" proof; usually a lemma and its proof are part of a larger proof of a theorem

A Corollary is a small proposition which follows rather immediately from a theorem or its proof.

A Proposition is just a (quantifiable) statement which is true or false; usually does not require much proof
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Modus ponens:

- If the statement "p implies q" is true, and the hypothesis p holds, then we know q must hold.

Proofs generally take the form: Assume p (which may not always hold), derive the conclusion q.

Whenever p, then q.

p is sufficient for q; q is necessary for p.

Sets
- A set is any (unordered) collection of (mathematical) objects; note one can have sets of sets (examples to follow).

Statements:
- An (open) statement: any mathematical statement; not necessarily true or false.
- Propositions: A statement which is true or false, and can be proved or disproved.

Remember Godel...
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Set builder
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  - $\{2, 4, \ldots\}$ B A D
- Set builder
  - $\{x : x > 0\}$
  - $\{x \mid x^2 + 5x + 6 = 0\}$
  - $\{z \mid z = 2 \times m + 1, \text{ where } m \text{ is an integer}\}$
Subset/superset:

- $S \subseteq R$ if all members of $S$ are also present in $R$ (proper subset: $S \subset R$ or $S \subsetneq R$)

- Universe: $U$ is the set from which all sets in considerations are drawn; if we are interested in a set $S$ then $S \subset U$ (context dependent)

- When talking about numbers, $U$ usually equals $C$, $R$, $Z$, or $N$

- Complement: For a set $S \subset U$ its complement is everything in $U$ not in $S$

  $$S' = U - S = U \setminus S$$

  Sometimes this is written as $S^c$.

- $x \in S$ is an element of $S$

  - if $x \not\in S$, then $x \in S'$.
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where “−” and “\( \setminus \)” are set minus (more later)
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Sometimes this is written as \( S^C \)

\( x \) is an element of \( S \) is written \( x \in S \),

if \( x \notin S \), then \( x \in S' \).
Let \( S, R \subset \mathcal{U} \).

- Set Equality: \( S = R \) if
  - \( S \) and \( R \) have exactly the same elements
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- **Difference:** \( S - R = \{ x \in S : x \notin R \} = S \setminus R \)

- **Symmetric Difference:** \( S \triangle R \equiv \{ x \in \mathcal{U} : x \in S \text{ and } x \notin R \text{ or } x \notin S \text{ and } x \in R \} \)
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Webster Proof and Its History
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- There exists $\exists$
- There exists a unique $\exists !$
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- Cardinality: $|S| = \#(S)$ number of elements in $S$.
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