Section 2.1

Problem 1

Yes, something can be both an element and a subset of the same set. For example, \{
1
\} is both an element and a subset of the set \{1,\{1\}\}.

Problem 3

\(A \times \emptyset = \emptyset \times A = \emptyset\).

Problem 4

(a) \{{1}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, \{1, 2, 3, 4\}\}.

(b) \{{1,2,3}, \{1,2,4\}, \{1,2,5\}, \{1,3,4\}, \{1,3,5\}, \{1,4,5\}, \{2,3,4\}, \{2,3,5\},
\{2,4,5\}, \{3,4,5\}\}.

(c) \{((1), (1)), ((1), (1, 2)), ((2), (2)), ((2), (1, 2)), ((1, 2), (1, 2))\}.

Problem 5

(a) False. Let \(A = \{1, 2\}, B = \{1, 3\},\) and \(C = \{2, 3\}.\) Then \(A \cup B = \{1, 2, 3\},\)
\(B \cap C = \{3\}.\) Thus 
\((A \cup B) \cap C = \{2, 3\}\)
but 
\(A \cup (B \cap C) = \{1, 2, 3\}.
\)

A correct formulation is 
\((A \cup B) \cap C = (A \cap C) \cup (A \cap B).\)
(b) False. Let the universe \( U = \{1, 2, 3, 4\} \), let \( A = \{1, 2\} \), and let \( B = \{2, 3\} \). Then \( A \cap B = \{2\} \), \( A' = \{3, 4, 5\} \), and \( B' = \{1, 4\} \). Thus

\[
(A \cap B)' = \{1, 3, 4\}
\]

but

\[
A' \cap B' = \{4\}.
\]

A correct formulation is

\[
(A \cap B)' = A' \cup B'.
\]

(c) False. Let \( A = \{1, 2\} \), \( B = \{2, 3\} \), and \( C = \{1, 3\} \). Then \( B \cup C = \{1, 2, 3\} \), \( A - B = \{1\} \), and \( A - C = \{2\} \). Thus

\[
A - (B \cup C) = \emptyset
\]

but

\[
(A - B) \cup (A - C) = \{1, 2\}.
\]

A correct formulation is

\[
A - (B \cup C) = (A - B) \cap (A - C).
\]

(d) True.
Problem 8

(a) Vacuously true. It is impossible for two sets to be proper subsets of one another.

(b) True. If $A$ is a subset of $A \cap B$, then $A$ “fits inside” their intersection, then it “fits inside” both $A$ and $B$.

(c) False. Let $A = \{1, 2\}$, $B = \{1, 2\}$, and $C = \{2, 3\}$. Then $A \cup B = \{1, 2\} \subseteq A \cup C = \{1, 2, 3\}$, but $B \not\subseteq C$.

(d) False. Let $A = \{1, 2\}$, $B = \{2, 3\}$, $C = \{1, 2, 3\}$, and $D = \{2, 4\}$. Then $A \cap B = \{2\} \subseteq \{2\} = C \cap D$, but $B \not\subseteq D$.

(e) False. Let $A = \{1, 2\}$, $B = \{2, 3\}$, and $C = \{1, 2, 4\}$. Then $A \not\subseteq B$ and $B \not\subseteq C$, but $A \subseteq C$.

(f) Vacuously true. A power set is a set of sets, not the empty set.
(g) False. Let \(A = \{1\} \), \(B = \\{\{1\}, 2\} \), and \(C = \\{\{1\}, 2\} \). Then \(A \in B \) and \(B \in C \), but \(A \notin C \).

(h) False. Let \(A = \{1\} \), \(B = \{1, 2\} \), and \(C = \{1, 2, 3\} \). Then \(A \subseteq B \) and \(B \in C \), but \(A \notin C \).

(i) True. If \(A \) “fits inside” \(B \), then everything that’s not in \(B \) should “fit inside” everything that’s not in \(A \).

(j) False. Let \(A = \emptyset \), \(B = \{3\} \), \(C = \{1\} \), and \(D = \{1, 2\} \). Then \(A \times B = \emptyset \subseteq C \times D = \{(1, 1), (1, 2)\} \), but \(B \not\subseteq D \).

(k) True. In this case we don’t have to worry about the empty set messing things up, since the two sets in the Cartesian products are the same: even if \(A = \emptyset \), \(A \times A = \emptyset \) is still a subset of \(B \times B \).

(l) True. If \(A \) “fits inside” \(B \), then it doesn’t intersect anything outside of \(B \).

**Problem 10**

(a) \(\{1, 4, 9, 16, 25, 36, 49\} \).

(c) \(\{4, -1\} \).

(e) \(\{3, 8, 13, 18, 23\} \).

(f) \(\{\{2, 4\}, \{1, 2, 4\}, \{2, 3, 4\}, \{1, 2, 3, 4\}\} \).

**Problem 12**

(a) \(|A \cup B| = s + t - u| \).

(b) \(|A \times B| = st| \).

(c) \(|P(A)| = 2^s| \).

(d) \(|A - B| = s - u| \).

**Problem 14a**

(i) False. Let \(A = \{1, 2\} \), \(B = \{2, 3\} \), and \(C = \{1, 3\} \). Then \(A \triangle (B \cup C) = (A - (B \cup C)) \cup ((B \cup C) - A) = ((1, 2) \setminus \{1, 2, 3\}) \cup (\{1, 2, 3\} \setminus \{1, 2\}) = \emptyset \cup \{3\} = \{3\} \), while \((A \triangle B) \cup (A \triangle C) = ((A - B) \cup (B - A)) \cup ((A - C) \cup (C - A)) = (\{1\} \cup \{2\}) \cup (\{2\} \cup \{3\}) = \{1, 2, 3\} \).

A correct formulation is \(A \triangle (B \cup C) = (A \triangle B) \cup C - (A \cap C) \). (There are many others.)
(ii) False. Let \( A, B, \text{ and } C \) be as in part (i). Then 
\[
A \triangle (B \cap C) = (A - (B \cap C)) \cup ((B \cap C) - A) = (\{1, 2\} - \{3\}) \cup (\{3\} - \{1, 2\}) = \{1, 2\} \cup \{3\} = \{1, 2, 3\},
\]
while 
\[
(A \triangle B) \cap (A \triangle C) = ((A - B) \cup (B - C)) \cap ((A - C) \cup (C - A)) = (\{1\} \cup \{2\}) \cap (\{2\} \cup \{3\}) = \{2\}.
\]
A correct formulation is 
\[
A \triangle (B \cap C) = ((A \triangle B) \cap C) \cup (A - B).
\]
(There are many others.)

(iii) True.

(iv) False. If \( A = \{1\}, \) then 
\[
A \triangle A = (A - A) \cup (A - A) = \emptyset \cup \emptyset = \emptyset \neq A.
\]
A correct formulation is \( A \triangle A = \emptyset. \)

**Problem 6 (bonus)**

(a) False. Let \( A = \{1, 2\} \) and \( B = \{2, 3\}. \) Then \( \{1, 3\} \in \mathcal{P}(A \cup B) \) since it is a subset of \( A \cup B = \{1, 2, 3\}, \) but \( \{1, 3\} \notin \mathcal{P}(A) \cup \mathcal{P}(B) \) since it is not a subset of either \( A \) or \( B. \)

(b) True. Any element of \( \mathcal{P}(A \cap B) \) is a subset of \( A \cap B, \) and so it has to be a subset of \( A \) and a subset of \( B, \) and thus is in \( \mathcal{P}(A) \cap \mathcal{P}(B). \) Conversely, any element of
\( \mathcal{P}(A) \cap \mathcal{P}(B) \) is a subset of both \( A \) and \( B \), and so it is a subset of \( A \cap B \), and thus an element of \( \mathcal{P}(A \cap B) \).

(c) True. Any element of \( A \times (B \cup C) \) has its first coordinate in \( A \) and its second coordinate in either \( B \) or \( C \). So it will certainly be in either \( A \times B \) or \( A \times C \). Conversely, any element of \((A \times B) \cup (A \times C)\) either in \( A \times B \) or \( A \times C \), so its first coordinate will be in \( A \) and its second coordinate will be in either \( B \) or \( C \). (If one side of the equation is the empty set, you can check that the other side must be the empty set too.)

(d) True. Any element of \( A \times (B \cap C) \) has its first coordinate in \( A \) and its second coordinate in both \( B \) and \( C \). So it will certainly be in \( A \times B \) and \( A \times C \). Conversely, any element of \((A \times B) \cap (A \times C)\) is in both \( A \times B \) and \( A \times C \), so that its first coordinate is in \( A \) and its second coordinate is in both \( B \) and \( C \). (If one side of the equation is the empty set, you can check that the other side must be the empty set too.)

(e) False. Let \( A = \{1\} \) and \( B = \{2\} \). Then \( \mathcal{P}(A \times B) = \mathcal{P}(\{(1,2)\}) = \{\emptyset, \{(1,2)\}\} \), while \( \mathcal{P}(A) \times \mathcal{P}(B) = \{\emptyset, \{1\}\} \times \{\emptyset, \{2\}\} = \{(\emptyset, \emptyset), (\emptyset, \{2\}), (\{1\}, \emptyset), (\{1\}, \{2\})\} \).

Problem 13 (bonus)

For 0 sets, the Venn diagram is just a box, so there is 1 portion.
For 1 set, the Venn diagram has 2 portions.
For 2 sets, the Venn diagram has 4 portions.
For 3 sets, the Venn diagram has 8 portions.

The pattern seems to be: for \( n \) sets, the Venn diagram has \( 2^n \) portions. We can see why this might be true by considering what happens when we add a new set to our consideration. Suppose we have \( n \) sets in a Venn diagram (so that the diagram has \( 2^n \) portions) and we add an \( n+1 \)st one, \( S_{n+1} \). For each of the \( 2^n \) original portions, either \( S_{n+1} \) intersects it or it doesn’t - that is, we have two distinct choices. When we build the new diagram, then, we will need \( 2 \times 2^n = 2^{n+1} \) portions.

Problem 14b (bonus)

\[ |A \triangle B| = |A| - |A \cap B| + |B| - |A \cap B| = s + t - 2u. \]