10/28/2015

Graded Activity

MATH 105, Fall 2015

Name (Please print):________________________________

Answer the following questions, showing all work where appropriate **on other sheets.** You can work with others, but you must turn in your own work.

1. Find the inflection point(s) (x values) of the following two curves, making sure to justify the change in concavity at each value[1]:

   (a) $f(x) = (x-1)x(x+2)$

   (b) $f(x) = x^4 - 2x^2 - 1$

2. Consider a function $g(x)$ so that $g'(x) = (x-1)(x-2)(x-3)$.

   (a) What are the critical values of $g(x)$?

   (b) Draw a number line which gives the *sign* of $g'(x)$ on either side of each critical point.

   (c) Use the *first derivative test* to determine the x values corresponding to local extrema (don't try to find the y-values). Justify your answers.

   (d) Suppose that $g(1) = -2.25$, $g(2) = -2$, $g(3) = -2.25$, $g(-1) = 13.75$, $g(0) = 0$, and $g(4) = 0$. Compute the global extrema of $g(x)$ for $-1 \leq x \leq 4$.

   (e) Compute $g''(x)$.

   (f) Compute $g''(c)$ for each critical value c you found in the previous part.

   (g) Use the *second derivative test* to determine what type of local extremum each critical value is. Justify your answers.

   (h) Which of the *derivative tests* do you like better? Explain.

3. Suppose that a represents the amount of fertilizer (in some sensible units) applied to a field of pumpkins. Then the function $Y(a)$ represents the *yield* of pumpkins. Suppose you plot your data from the last few years and decide

$$Y(a) = -(.05)a^3 + 10a^2 + 100, \quad 0 \leq f \leq 100.$$

   Compute the point of diminishing returns (this is an a value). Then explain what this value *means* in practical terms.

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[1] Compute the *hypercritical values* of $g(x)$. Draw a number line which gives the *sign* of $g''(x)$ on either side of each hypercritical point.

4. The manufacturer of TVs can sell 800 sets to his dealers at \$392 each. If instead the price is \$380, he can sell 1000 sets.

   (a) Assume the demand function $p(x)$ is a linear function and use the data points $(800, 392)$ and $(1000, 380)$ to compute the linear equation for $p(x)$ (the price of the units as a function of the number of units).

   (b) Compute the revenue function $R(x)$ (using the fact that revenue equals the cost per set times the number of sets sold).

   (c) Assume the cost function is given by $C(x) = 3050 - 10x - .01x^2$. Write down and simplify the profit function $P(x)$.

   (d) Maximize the profit function $P(x)$ to determine the number of units $x = m$ which will yield the most profit. (Use either the first or second derivative test to do this and show all work.)

Graded Activity Solutions

1. a) $f(x) = (x-1)x(x+2)$

$$= (x^2 - x)(x+2)$$

$$= x^3 + 2x^2 - x^2 - 2x$$

$$= x^3 + x^2 - 2x$$

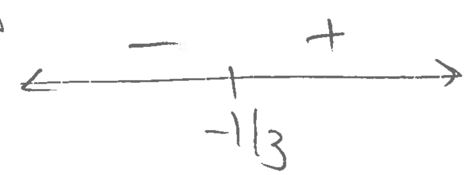
$$f'(x) = 3x^2 + 2x - 2$$

$$f''(x) = 6x + 2$$

Hypercritical value(s): when $f''(x) = 0$.

$x = -1/3$.

Plot: sign $f''(x)$



$f''(-1) = -$

$f''(0) = +$

Thus, since concavity of f changes at $x = -1/3$, $(-1/3, f(-1/3))$ is an inflection pt.

b) $f(x) = x^4 - 2x^2 - 1$

$f'(x) = 4x^3 - 4x$

$f''(x) = 12x^2 - 4$

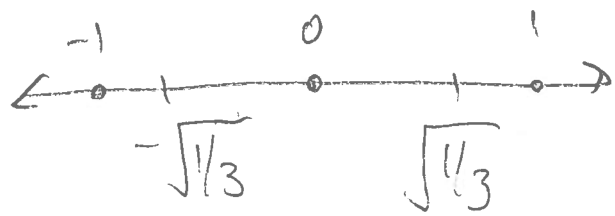
Set $f''(x) = 0$ + solve

$12x^2 - 4 = 0 \Rightarrow x^2 = 1/3$

$\Rightarrow x = \pm\sqrt{1/3}$

Plot sign of $f''(x)$ to see if concavity changes at the hyper critical values above.

sign f''



$f''(-1) = +$

$f''(0) = -$

$f''(1) = +$

Thus $\left(-\sqrt{1/3}, f(-\sqrt{1/3})\right)$

and $\left(\sqrt{1/3}, f(\sqrt{1/3})\right)$ are inflection points.

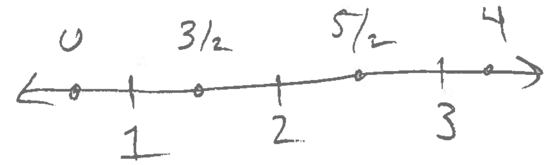
2. $g'(x) = (x-1)(x-2)(x-3)$

a) The critical values are where

$f'(x) = 0$ or DNE.

Thus $x = 1, 2, 3$.

b) sign of g'



$g'(0) = (-)(-)(-) = -$

$g'(3/2) = +(-)(-) = +$

$g'(5/2) = (+)(+)(-) = -$

$g'(4) = (+)(+)(+) = +$

c) Since g' changes sign at $x = 1, 2, 3$
all correspond to local extrema
(first deriv. test.)

$(1, g(1))$ is a local min

$\searrow \nearrow$

$(2, g(2))$ is a local max

$\nearrow \searrow$

$(3, g(3))$ is a local min

$\searrow \nearrow$

2. d) Thus to find global extrema we consider the list of critical points and endpoints for $g(x)$ on $-1 \leq x \leq 4$

$$\{(1, g(1)), (2, g(2)), (3, g(3)), (-1, g(-1)), (4, g(4))\}$$

$$= \{(1, -2.25), (2, -2), (3, -2.25), (-1, 13.75), (4, 0)\}$$

The global max is $y = 13.75$ occurring at $x = -1$.

The global min is $y = -2.25$ occurring at $x = 1$ and 3.

e)
$$g'(x) = [(x-1)(x-2)(x-3)]'$$
$$= [(x^2 - 3x + 2)(x-3)]'$$
$$= [x^3 - 3x^2 - 3x^2 + 9x + 2x - 6]'$$
$$= (x^3 - 6x^2 + 11x - 6)'$$
$$= 3x^2 - 12x + 11.$$

2. f) $g''(1) = 2$

$g''(2) = 3 \cdot 4 - 12 \cdot 2 + 11$
$= -1$

$g''(3) = 3 \cdot 9 - 12 \cdot 3 + 11$
$= 27 - 36 + 11$
$= 2$

g) Since $g''(1) > 0$, g is concave up @ $x=1$, so $(1, g(1))$ is a local min.

Since $g''(2) < 0$, g is concave down @ $x=2$, so $(2, g(2))$ is a local max.

$(3, g(3))$ is a local min by the same reasoning.

h) Personal preference.

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3. The point of diminishing returns occurs (for a factor of production vs. output production model) when the rate of return per unit input begins to decrease. This is the value of the factor of production for which the rate of return is maximized. In calculus terms, this is the inflection pt.

So for a = amount of fertilizer applied

$Y(a)$ = yield of pumpkins

we are looking for an inflection pt of

$$Y(a) = -.05a^3 + 10a^2 + 100 \qquad a \in [0,100]$$

$$\frac{dY}{da} = (-.15)a^2 + 20a \qquad \frac{d^2Y}{da^2} = (-.3)a + 20$$

set $Y''(a) = 0$ and solve: $a = \frac{20}{.3} = \frac{200}{3}$

3 cont.

$$y''(a) = -.3a + 20$$

$$y''(0) = 20 > 0, \quad y''(100) = -10$$

Thus $(200/3, \; y(200/3))$

$= (66.67, \; 29729.6)$

is the point of diminishing returns, where $y'(a)$ is maximized

$$y'(200/3) = 666.67.$$

$a = 66.67$ is the amount of fertilizer which yields the highest per unit return on the number of pumpkins yielded.

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4. a) $p(x)$ is linear

$(800, p(800)) = (800, 392)$

$(1000, p(1000)) = (1000, 380)$

$(p(x) - y_0) = m(x - x_0)$

$m = \frac{\text{rise}}{\text{run}} = \frac{380-392}{1000-800} = \frac{-12}{200} = \frac{-3}{50}$

$p(x) - 380 = \frac{-3}{50}(x - 1000)$

$p(x) = -\frac{3}{50}x + 480$

b) $R(x) = x \quad p(x)$

units sold price/unit

$= x\left(-\frac{3}{50}x + 480\right)$

$= -\frac{3}{50}x^2 + 480x$

c) $C(x) = 3050 - 10x - .01x^2$

$P(x) = R(x) - C(x)$

$= -\frac{3}{50}x^2 + 480x - [3050 - 10x - .01x^2]$

4 c.) cont 9

$P(x) = -.05x^2 + 490x - 3050$

$P'(x) = -.05(2)x + 490$

$= -.1x + 490$

set $= 0$

$0 = P'(x) = -.1x + 490$

$x = 4900$

$P''(x) = -.1 \quad < 0$

$P(x)$ always concave down

so $x = 4900$ corresponds to a local + global max. by the 2^{nd} deriv. test.

Thus $(4900, P(4900))$ is the global max

$(4900, 1.2 \times 10^6)$