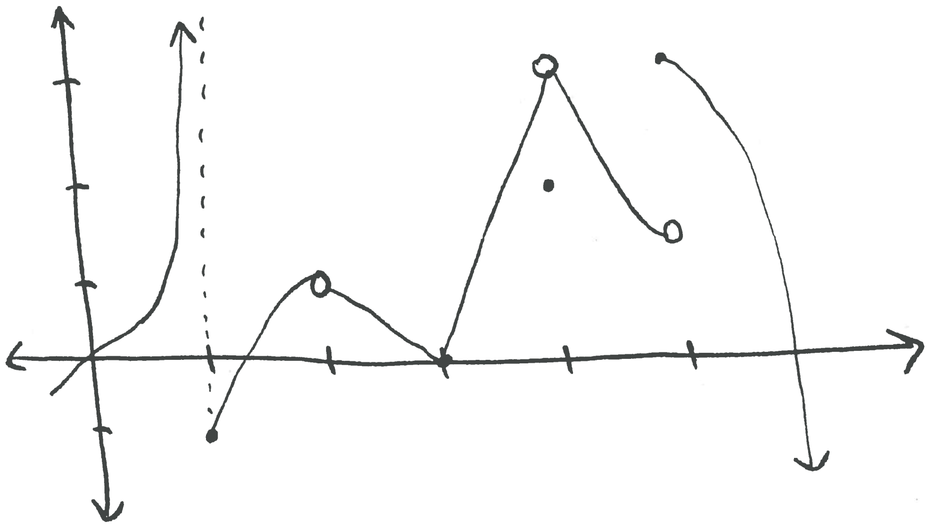
MATH 105 - WEBSTER TEST 1 MARCH 2ND, 2017

NAME: KEY

Please read the questions. Show all of your reasoning and calculations (the more you show, the more I can give you points for). Try to be organized and clear. Solutions/answers without work are not worth many points. You do not need to simplify your answers unless explicitly told to do so. This is a closed note/closed book test. Calculators capable of symbolic manipulation are not allowed. GOOD LUCK!

Problem	Points	Score
1	5	
2	12	
3	12	
4	10	
5	19	
6	12	
7	10	
Bonus	+5	
Total	80	

1. (5 pts) Evaluate the following limits using the graph of $f(x)$ below.



(a) $\lim_{x \to 1} f(x)$ DNE (asymptote / jump)

(b) $\lim_{x \to 2} f(x) = 1$ (though $f(2)$ DNE)

(c) $\lim_{x \to 3} f(x) = 0$

(d) $\lim_{x \to 4} f(x) = 3$ (though $f(4) = 2$)

(e) $\lim_{x \to 5} f(x)$ DNE (jump)
(though $f(5) = 3$)

2. (12 pts) Compute the following limits.

(a) $\lim_{x\to 0} \frac{x^2-3x+2}{x^2-x-2} = \frac{0^2-3\cdot 0+2}{0^2-0-2} = \frac{2}{-2} = \boxed{-1}$

just plug in

(b) $\lim_{x\to 1} \frac{x^2-3x+2}{x^2-x-2} = \lim_{x\to 1} \frac{(x-2)(x-1)}{(x-2)(x+1)} = \frac{-1\cdot 0}{-2\cdot 2} = \boxed{0}$

just plug in

(c) $\lim_{x\to 2} \frac{x^2-3x+2}{x^2-x-2} = \lim_{x\to 2} \frac{\cancel{(x-2)}(x-1)}{\cancel{(x-2)}(x+1)} = \lim_{x\to 2} \frac{x-1}{x+1} = \boxed{\frac{1}{3}}$

properties of limit

(d) $\lim_{x\to -1} \frac{x^2-3x+2}{x^2-x-2} = \lim_{x\to -1} \frac{(x-2)(x-1)}{(x-2)(x+1)} = "\frac{-2}{0}"$ $\boxed{\text{DNE}}$

vertical asymptote

(e) $\lim_{\alpha\to 3} \left[\left(\frac{\sqrt{\alpha+6}}{\alpha-2}\right)((\alpha-2)^2+1)\right] = \frac{\sqrt{9}}{1}\cdot(1^2+1) = \boxed{6}$

(f) $\lim_{h\to 0} \frac{\cancel{h}(h+x+hx^2)}{\cancel{h}} = \lim_{h\to 0} h+x+hx^2 = 0+x+0\cdot x^2$
$= \boxed{x}$

3. (12 pts) Compute the derivatives of the following functions.

(a) (4 pts) $f(x) = 2x^{1.1} + 5^{2/3} + \frac{1}{x}$

$f'(x) = (1.1)\cdot 2\, x^{.1} + 0 - 1\, x^{-2}$

$= 2.2\, x^{.1} - x^{-2}$

power rule, sum rule

(b) (4 pts) $g(r) = \sqrt{5r} = (5r)^{1/2} = 5^{1/2} \cdot r^{1/2}$

$\frac{dg}{dr} = 5^{1/2} \cdot \frac{1}{2} r^{-1/2}$

constant mult. rule

or $= \frac{1}{2}(5r)^{-1/2} \cdot 5$

chain rule

(c) (4 pts) $y = \frac{x}{1 + x^{1/3}}$

$\frac{dy}{dx} = \frac{(1+x^{1/3})(1) - x\left(0 + \frac{1}{3}x^{-2/3}\right)}{(1+x^{1/3})^2}$ ✓

quotient

$= \frac{1 + x^{1/3} - \frac{1}{3}x^{1/3}}{(1+x^{1/3})^2}$

$= \frac{1 + \frac{2}{3}x^{1/3}}{(1+x^{1/3})^2}$

4. (10 pts)

(a) (4 pts) Let $f(x) = \left(\frac{3x+1}{1-x}\right)^{1/4}$. Compute the slope of the tangent line to $f(x)$ at $x = 0$.

$$m_{tan}\Big|_{x=0} = f'(0)$$

$$f'(x) = \frac{1}{4}\left(\frac{3x+1}{1-x}\right)^{-3/4} \cdot \left[\frac{3x+1}{1-x}\right]' \quad \text{chain}$$

$$= \frac{1}{4}\left(\frac{3x+1}{1-x}\right)^{-3/4}\left[\frac{(1-x)(3)-(3x+1)(-1)}{(1-x)^2}\right]$$

$$f'(0) = \frac{1}{4}(1)^{-3/4}\left[\frac{3+1}{1^2}\right] = \boxed{1}$$

(b) (6 pts) Compute the tangent line to the function $g(x) = x(1-2x)^3$ at the point $x = 1$.

Tangent given by $y - y_0 = m_{tan}\Big|_{x=1}(x-1)$

$$y_0 = g(1) = 1(-1)^3 = -1$$

$$g'(x) = 1\cdot(1-2x)^3 + x\left[3(1-2x)^2\cdot(-2)\right] \quad \text{product + chain}$$

$$= (1-2x)^3 - 6x(1-2x)^2$$

$$y'(1) = -1 - 6(1) = -7$$

$$\boxed{y = -7(x-1) - 1} \text{ or } \boxed{y-(-1) = -7(x-1)}$$

5. (19 pts) You are about to open a business to manufacture widgets. Let x be the number of widgets produced. You decide to price them according to a demand function $p(x) = 50 - x$ dollars. It costs you \$20 per unit to produce widgets, with a fixed cost of \$200.

(a) (2 pts) Compute the cost function $C(x)$.

$$C(x) = 20x + 200$$

(b) (2 pts) Compute the revenue function $R(x)$.

$$R(x) = x\,p(x) = x(50 - x) = 50x - x^2$$

(c) (1 pts) Compute the profit function $P(x)$.

$$P(x) = R(x) - C(x) = -x^2 + 50x - (20x + 200)$$
$$= -x^2 + 30x - 200$$

(d) (2 pts) Compute the *break even points.*

$$P(x) = -(x-10)(x-20)$$

$\boxed{\text{BEP: } x = 10, 20}$

(e) (3 pts) Compute the *marginal average cost* function $(\overline{C})'(x)$.

$$\overline{C}(x) = \frac{C(x)}{x} = 20 + \frac{200}{x}$$
$$(\overline{C})'(x) = (200x^{-1})' = -200x^{-2}$$

(f) (2 pts) Compute $\dfrac{dP}{dx}$.

$$\frac{dP}{dx} = -2x + 30$$

(g) (2 pts) Explain the meaning of $\left.\frac{dP}{dx}\right|_{x=12}$, i.e., its interpretation in this context. (Should the 13th widget be made?)

$$\left.\frac{dP}{dx}\right|_{x=12} = -2(12)+30 = 6$$

The marginal profit is >0, so $P(x)$ increasing at $x=12$. The 13th widget should be made.

(h) (4 pts) Now assume that the number of units produced, x, depends on the day in the week in the following way:

$$x(t) = t(7-t), \quad \text{where } 0 \leq t \leq 7.$$

With your formula for $P(x)$ as above, compute $\frac{dP}{dt}$, and express your answer as a function of t.

$$\frac{dP}{dx} = -2x+30 \qquad \frac{dx}{dt} = 7-2t$$

$$\frac{dP}{dt} = (-2x+30)(7-2t) = (-2(t(7-t))+30)(7-2t)$$

$$\frac{dP}{dt} = (-14t+2t^2)(7-2t)$$

(i) (1 pts) Compute $\left.\frac{dP}{dt}\right|_{t=3}$ and explain what this quantity represents.

$$\left.\frac{dP}{dt}\right|_{t=3} = (-14\cdot 3+2\cdot 9)(1)$$

$$= -24$$

This is the r.o.c. of the profit with respect to day of the week on the third day.

6. (12 pts) Let $f(t) = t^2 + t$ and consider the domain $t \in [0, 2]$.

(a) (2 pt) The average rate of change on $[0, 2]$ can best be visualized as the <u>slope</u> of the <u>secant</u> line connecting the points (<u>0</u> , <u>0</u>) and (<u>2</u> , <u>6</u>).

(b) (2 pt) The instantaneous rate of change at $t = 1$ can best be visualized as the <u>slope</u> of the <u>tangent</u> line at the point (<u>1</u> , <u>2</u>).

(c) (2 pt) Compute the average rate of change of $f(t)$ on $[0, 2]$.

$$\frac{f(2)-f(0)}{2-0} = \frac{6-0}{2-0} = \boxed{3}$$

(d) (3 pts) Compute the average rate of change (depending on h) of $f(t)$ on $[1, 1+h]$ (simplify your answer).

$$\frac{f(1+h)-f(1)}{1+h-1} = \frac{(1+h)^2+(1+h)-[1^2+1]}{h}$$
$$= \frac{1+h^2+2h+1+h-2}{h}$$
$$= \frac{1}{h}[h^2+3h]$$

(e) (3 pts) Compute the instantaneous rate of change for $f(t)$ at $t = 1$ by taking an appropriate limit.

$$f'(1) = \lim_{h\to 0} \frac{h^2+3h}{h} = \lim_{h\to 0} \frac{h(h+3)}{h}$$
$$= \lim_{h\to 0} h+3 = \boxed{3}$$

7. (10 pts) Consider the function $f(x) = -\frac{1}{4}x^4 + x^3 - x^2$

(a) (3 pts) Compute $f'(x)$.

$$f'(x) = -4\left(\tfrac{1}{4}\right)x^3 + 3x^2 - 2x$$
$$= -x^3 + 3x^2 - 2x$$
$$= -x(x^2 - 3x + 2) = -x(x-2)(x-1)$$

(b) (3 pts) Compute the *critical values* of $f(x)$.

Set $f'(x) = 0$. (or DNE — not pertinent here)

$$0 \overset{\text{set}}{=} -x(x-2)(x-1)$$

$x = 0, 1, 2$ are CVs

(c) (4 pts) Determine the interval(s) where $f(x)$ is increasing, and the interval(s) where $f(x)$ is decreasing.

Sign of f'

$+ \quad - \quad + \quad -$

$0 \quad 1 \quad 2$

$f'(-1) = (+)(-)(-)$

$f'(1/2) = (-)(-)(-)$

$f'(3/2) = (-)(-)(+)$

$f'(3) = (-)(+)(+)$

f is increasing on $(-\infty, 0) \cup (1, 2)$ since $f' > 0$ on those intervals.

f is decreasing on $(0, 1) \cup (2, \infty)$ by the same logic.

Bonus (+5 pts) For a given function $f(x)$, compare and contrast the following objects:

- $f(1)$
- $f'(1)$
- $[f(1)]'$