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Chain Rule Application

MATH 105, Spring 2017

Chain Rule and Leibniz Notation

We begin by recalling the chain rule: for two functions $f(x)$ and $g(x)$ we can *compose* them. This is to say we can form the composite function $f \circ g(x) = f(g(x))$. Informally, this means "plug" in the function $g(x)$ to the function $f(x)$.

In doing this, we can identify the "outer" and "inner" functions, $f(x)$ and $g(x)$ respectively. Then, if we want to differentiate the composite $f(g(x))$, the mantra we have is: "derivative of outer, evaluated at inner, times derivative of inner." Mathematically:

$$\frac{d}{dx}\left[f(g(x))\right] = f'\left(g(x)\right) g'(x).$$

Notice, we have mixed Leibniz and Lagrange notation. In some situations (like the problem discussed below) it is easier to use Leibniz notation on the RHS as well.

Consider the following situation: if we have a composite function $y = f(g(x))$, we could write $y = f(u)$ where $u = g(x)$. In this case we have deconstructed our composition into its constituent pieces (inner and outer functions). By doing so explicitly, we can write the chain rule in a "nice" way:

$$\frac{dy}{dx} = \left.\frac{dy}{du}\right|_{u=g(x)} \frac{du}{dx}.$$

For short we just write:

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx},$$

which is nice because it shows that Leibniz notation allows us to treat derivatives like fractions in some sense; the chain rule is just what we'd expect if we "cancel out" the du from numerator and denominator.

In any case, it's the same rule just written slightly differently. We'll use it now below.

Business Application

> A manufacturer has determined that the weekly profit from the sale of x items is given by the function below. It is estimated that after t days in any week, x items will have been produced. Find the rate of change of profit with respect to time at the end of 6 days.
>
> $$P(x) = -2x^2 + 800x - 2000 \text{ with } x = 2.5t^2 - 3t$$

Let us begin to solve this problem in a more standard way. Since P depends on x (i.e., $P(x)$) and x depends on t (i.e., $x(t)$), what we really have is a composition $P(x(t))$. So let us plug in $x(t) = 2.5t^2 - 3t$ into the function $P(x) = -2x^2 + 800x - 2000$:

$$P(x(t)) = -2x^2 + 800x - 2000 = -2[2.5t^2 - 3t]^2 + 800[2.5t^2 - 3t] - 2000.$$

Since the problem is asking for the "rate of change of profit at the end of six days" we need the instantaneous rate of change of P at $t = 6$. This is to say, we need $\left.\frac{dP}{dt}\right|_{t=6}$. Without attempting to simplify this algebraically, we can differentiate using the chain rule:

$$\begin{aligned}\frac{dP}{dt} &= 2(-2)[2.5t^2 - 3t]^1[2(2.5)t^1 - 3] + 800[2(2.5)t - 3] \\ &= (-4)[2.5t^2 - 3t][5t - 3] + 800[5t - 3].\end{aligned}$$

Thus, plugging in $t = 6$ yields:

$$\begin{aligned}\left.\frac{dP}{dt}\right|_{t=6} &= (-4)[2.5(6)^2 - 3(6)][5(6) - 3] + 800[5(6) - 3] \\ &= 13824.\end{aligned}$$

Now, plugging in $x(t)$ to $P(x)$ directly is not the only way to do this problem. Since P depends on x, but x depends on t, ultimately P depends on t. Since P depends on both quantities, there are two derivatives we could compute $\frac{dP}{dx}$ and $\frac{dP}{dt}$. (Note that using P' here would be misleading because we haven't specified what the independent variable is.) Moreover, one can certainly compute $\frac{dx}{dt}$. Thus, let us compute these.

$$\begin{aligned}\frac{dP}{dx} &= (-4)x + 800 \\ \frac{dx}{dt} &= 5t - 3.\end{aligned}$$

In Leibniz notation, the chain rule says

$$\frac{dP}{dt} = \frac{dP}{dx}\frac{dx}{dt} = [-4x + 800][5t - 3] = [-4(2.5t^2 - 3t) + 800][5t - 3].$$

Thus, to compute $\left.\frac{dP}{dt}\right|_{t=6}$ we simply plug in $t = 6$ in the previous calculation:

$$\left.\frac{dP}{dt}\right|_{t=6} = [-4(2.5(6)^2 - 3(6)) + 800][5(6) - 3] = 13824.$$

Either approach is equivalent!