3/16/2017

Activity 2 Solutions

MATH 105, Spring 2017

This a graded activity (graded for completeness and thoroughness, though not correctness). **Complete all work on separate sheets of paper, showing all details and answering in complete sentences.** You may work in groups, but please turn in your own paper. If you do not finish in class, finish it at home and turn in to my office by the end of the day on Monday, 3/20.

1. Suppose that $f(x)$ is a function with derivative

$$f'(x) = \frac{x(x-2)^2(x-1)}{(x+1)^3(x+2)}.$$

(a) Compute the critical values of $f(x)$.

Critical values occur whenever $f'(x) = 0$ or does not exist. We also require that such x-values are in the domain of of the original function, but since that function is not given, we will proceed and ignore that requirement.

The derivative is already factored, and thus we are looking for zeros (when the denominator is zero) or points where $f'(x)$ DNE: it's clear that the zeros are $x = 0, 1, 2$ and that $x = -1, -2$ are not in the domain of $f'(x)$. Thus we have the values: $x = -2, -1, 0, 1, 2$.

(b) Make a sign chart for $f'(x)$. We'll choose test points $x = -3, -1.5, -.5, .5, 1.5, 3$.

$$\begin{aligned}
f'(-3) &= \frac{(-)(-)^2(-)}{(-)^3(-)} = \frac{(-)(+)(-)}{(-)(-)} = +\\
f'(-1.5) &= \frac{(-)(-)^2(-)}{(-)^3(+)} = \frac{(-)(+)(-)}{(-)(+)} = -\\
f'(-.5) &= \frac{(-)(-)^2(-)}{(+)^3(+)} = \frac{(-)(+)(-)}{(+)(+)} = +\\
f'(.5) &= \frac{(+)(-)^2(-)}{(+)^3(+)} = \frac{(+)(+)(-)}{(+)(+)} = -\\
f'(1.5) &= \frac{(+)(-)^2(+)}{(+)^3(+)} = \frac{(+)(+)(+)}{(+)(+)} = +\\
f'(3) &= \frac{(+)(+)^2(+)}{(+)^3(+)} = \frac{(+)(+)(+)}{(+)(+)} = +
\end{aligned}$$

The chart for *sign of* f' can then be constructed using the test points above to determine the sign on the intervals between critical values.

(c) Describe the open sets where $f(x)$ is increasing and where $f(x)$ is decreasing.

The original funciton, $f(x)$, is increasing whenever $f' > 0$ and decreasing whenever $f' < 0$. Thus, f is increasing on:

$$(-\infty, -2) \cup (-1, 0) \cup (1, 2) \cup (2, \infty);$$

and f is decreasing on:

$$(-2, -1) \cup (0, 1).$$

(d) Find and classify the local extrema of $f(x)$ using your sign chart. (Present your answer in the form of complete sentences.)

The candidates for extrema are $x = 0, 1, 2$. There are sign changes for the derivative at $x = 0$ and $x = 1$, thus they correspond to extrema. Our conclusions, via the first derivative test: (i) $(0, f(0))$ is a local maximum, since f goes from increasing to decreasing there (f' goes $+ \to -$); (ii) $(1, f(1))$ is a local minimum, since f goes from decreasing to increasing there (f' goes $- \to +$). Note that $(2, f(2))$ is not an extrema, since the derivative does not change sign at that critical value.

2. Find and classify the local extrema of $f(x) = x^3 - 6x^2 + 8x - 2$, showing all work. (You'll need the QF here.) We begin by finding the derivative: $f'(x) = 3x^2 - 12x + 8$. This is a continuous function, so we need only find its zeros. We do so by setting it equal to zero and solving:

$$0 =^{\text{set}} f'(x) = 3x^2 - 12x + 8.$$

This does not factor nicely, so we'll invoke the Quadratic Formula:

$$x = \frac{-(-12) \pm \sqrt{(-12)^2 - 4(3)(8)}}{2 \cdot 3} = 2 \pm \frac{2\sqrt{3}}{3} \approx .85, 3.15$$

To determine intervals of increase/decrease (and thus create a sign chart) we need to test points on either side of the critical values. We'll choose $x = 0, 1, 4$.

$$f'(0) = 8 = +$$
$$f'(1) = \;-3 = -$$
$$f'(4) = 8 = +$$

Thus, by the first derivative test (see the previous answer for explanation), we note that $(.85, f(.85)) = (.85, 1.08)$ is a local max, and $(3.15, f(3.15)) = (3.15, -5.08)$ is a local min.

3. Find the absolute extrema of $f(x) = x^3 - 6x^2 + 8x - 2$ on each of the following intervals:

To find the absolute extrema, we need to check any critical values in the interval of consideration, as well as the endpoints. We will simply compare the y-values, and note that the largest is the absolute max on that interval, and the smallest y-value is the absolute min on that interval.

(a) $[0, 5]$

Here, we note that both critical values $x = .85, 3.15$ are inside the interval. Thus we need to compare the y-values for $x = 0, .85, 3.15, 5$. Doing so, we see that $(5, 13)$ is the absolute max (occurring at an endpoint), and $(3.15, -5.08)$ is the absolute min (occurring at a local min).

(b) $[-1, 6]$

Here, we note that both critical values $x = .85, 3.15$ are inside the interval. Thus we need to compare the y-values for $x = -1, .85, 3.15, 6$. Doing so, we see that $(6, 46)$ is the absolute max (occurring at an endpoint), and $(-1, -17)$ is the absolute min (occurring at an endpoint).

(c) $[4, 5]$

Here, we note that neither critical value occurs inside the interval. Thus we need only to compare the y-values for $x = 4, 5$. Doing so, we see that $(5, 13)$ is the absolute max (occurring at an endpoint), and $(4, -2)$ is the absolute min (occurring at an endpoint).

(d) $[-1, 4]$

Here, we note that both critical values $x = .85, 3.15$ are inside the interval. Thus we need to compare the y-values for $x = -1, .85, 3.15, 4$. Doing so, we see that $(.85, 1.08)$ is the absolute max (occurring at a critical value), and $(-1, -17)$ is the absolute min (occurring at an endpoint).

Hint: You'll need a calculator here, but these will help: $f(-1) = -17$; $f'(-1) = 23$; $f(0) = -2$; $f'(0) = 8$; $f(4) = -2$; $f'(4) = 8$; $f(5) = 13$; $f'(5) = 23$; $f(6) = 46$; $f'(6) = 44$

4. Consider the function $g(x) = x^4 - 2x^2 - x$.

(a) Compute $g''(x)$.

First, we compute $g'(x) = 4x^3 - 4x - 1$. We then compute $[g'(x)]' = g''(x) = 12x^2 - 4 = 4(3x^2 - 1)$.

(b) Compute the *hypercritical values* of $g(x)$, i.e., the x-values for which $g''(x) = 0$.

$$0 =^{\text{set}} f'(x) = 4(3x^2 - 1) \iff 3x^2 - 1 = 0 \iff x^2 = \frac{1}{3}.$$

Thus the hypercritical values of g (also, the critical values of g') are: $x = \pm\frac{\sqrt{3}}{3}$.

(c) Make a sign chart for $g''(x)$.

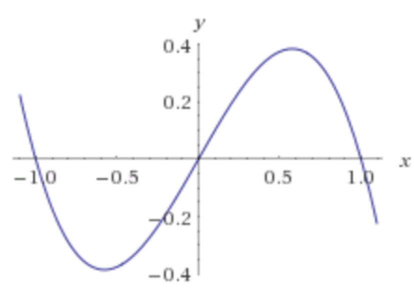
Since $\sqrt{3}/3$ is less than 1, we'll choose test points $x = -1, 0, 1$.

$$g''(-1) = +$$
$$g''(0) = -$$
$$g''(1) = +.$$

Thus g'' changes sign at both of the HCVs.

(d) Describe, with a picture, what the graph of $g(x)$ looks like when $g''(x)$ goes from $+ \to -$.

When $g'' > 0$, we can think of this in terms of what is happening to g'; namely, $g'' > 0$ is the same thing as $[g']' > 0$, and thus the first derivative is increasing (g' is increasing). This means that the slopes of the tangent lines to the graph of g are increasing, and we call this property *concave up*. Similarly, when $g'' < 0$, the slopes of the tangent line to g are decreasing in magnitude (g' is decreasing), and thus g is concave down.



Note that the above graph has this property (g'' goes $+ \to -$) at $x = 0$. Note that the graph of g is a "cup" on the LHS of $x = 0$ and a "cap" on the RHS of $x = 0$. We call $(0, g(0))$ a *point of inflection*.

5. Compute $\frac{d^2y}{dx^2}$ for the following functions:

(a) $y = x^2(2x+1)^{1/2}$

$$\begin{aligned}
\frac{dy}{dx} &= [x^2]'(2x+1)^{1/2} + x^2[(2x+1)^{1/2}]' \\
&= 2x(2x+1)^{1/2} + x^2\left(\frac{1}{2}(2x+1)^{-1/2}(2)\right) \\
&= 2x(2x+1)^{1/2} + x^2(2x+1)^{-1/2} \\
\frac{d^2y}{dx^2} &= 2(2x+1)^{1/2} + 2x(2x+1)^{-1/2} + 2x(2x+1)^{-1/2} - x^2(2x+1)^{-3/2} \\
&= 2(2x+1)^{1/2} + 4x(2x+1)^{-1/2} - x^2(2x+1)^{-3/2}
\end{aligned}$$

(b) $y = \frac{x}{1+x^3}$

$$\begin{aligned}
\frac{dy}{dx} &= \frac{(1+x^3)(1) - x(3x^2)}{(1+x^3)^2} \\
&= \frac{1-2x^3}{(1+x^3)^2} \\
\frac{d^2y}{dx^2} &= \frac{(1+x^3)^2(-6x^2) - (1-2x^3)(2(1+x^3)(3x^2))}{[(1+x^3)^2]^2} \\
&= \frac{6x^8 - 6x^5 - 12x^2}{(1+x^3)^4} \\
&= \frac{6x^5 - 12x^2}{(1+x^3)^3}
\end{aligned}$$